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Analytic approach to confinement and monopoles in lattice SU(2)

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Abstract: We extend the approach of Banks, Myerson, and Kogut for the calculation of the Wilson loop in lattice $U(1)$ to the non-abelian $SU(2)$ group. The original degrees of freedom of the theory are integrated out, new degrees of freedom are introduced in several steps. The centre group Z_2 enters automatically through the appearance of a field strength tensor $f_{\mu\nu}$, which takes on the values 0 or 1 only. It obeys a linear field equation with the loop current as source. This equation implies that $f_{\mu\nu}$ is non vanishing on a two-dimensional surface bounded by the loop, and possibly on closed surfaces. The two-dimensional surfaces have a natural interpretation as strings moving in euclidean time. In four dimensions we recover the dual Abrikosov string of a type II superconductor, i.e. an electric string encircled by a magnetic current. In contrast to other types of monopoles found in the literature, the monopoles and the associated magnetic currents are present in every configuration. With some plausible, though not generally conclusive, arguments we are directly led to the area law for large loops.

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1 Introduction

It is now widely accepted, that confinement is due to the formation of a color electric string, and that magnetic monopoles play an essential role in this context. Up to now there is a lively activity in this field, illuminating the phenomenon from various sides. A particularly appealing approach is the one by Banks, Myerson, and Kogut [1] which is now more than 20 years old. They considered the partition function for an electric current loop and derived step by step the appearance of monopoles by integrating out the original degrees of freedom and introducing new ones. The possibility to do this was restricted to the lattice U(1) model with the Villain action [2] and some other simple models. The authors also remark, that the techniques do not generalize simply to non-abelian theories.

We will start by applying and generalizing the methods of ref [1]. In a first step the $SU(2)$ matrices on the links are explicitly parametrized by three angles ψ, ϑ, φ. An appropriate decomposition of the $SU(2)$ matrices allows the calculation of the trace in the plaquette action. After an expansion of exponentials into modified Bessel functions the integrations over the link angles can be performed. They lead to constraints for the new variables which were introduced in the expansions. Most of these variables are irrelevant and the summations can be performed after a suitable transformation.

After this has been done, we are left with several integer variables which are restricted by constraint equations. The most important one is a Z_2 field strength tensor $f_{\mu\nu}$. It lives on plaquettes and is either 0 or 1. This tensor obeys an inhomogeneous linear field equation with the loop current as source. The solutions of the equation have a simple geometrical interpretation. The tensor $f_{\mu\nu}$ is non vanishing on a two-dimensional surface bounded by the loop, and possibly on closed two-dimensional surfaces. These surfaces have a natural interpretation as strings moving in euclidean time. There is a string which connects the two charges associated with the loop, and possibly a number of additional closed strings. The situation is particularly transparent for planar loops, where the layer on the minimal surface, corresponding to the straight string, plays a special role. For large loops we can use some general arguments and reasonings from statistical mechanics, like additivity of the free energy, and obtain the area law. The subtle question, whether a finite string tension survives in the thermodynamic limit and in the continuum limit, needs additional investigations.

Our approach is purely analytical and non-perturbative, no gauge fixing was performed, and β was kept arbitrary. Lengthy calculations were done with the help of Mathematica. Nowhere any physical picture of what we expect was put in. It is the formalism itself which automatically leads to the appearance of a plaquette variable $f_{\mu\nu}$ which is naturally associated with the world sheet of a string. If one could perform the integrations and summations over the remaining parameters one would obtain the explicit string action. Even without doing this, the formalism clearly shows the appearance of the string picture and the origin of the confinement mechanism.

2 The partition function

We are interested in the expectation value of a Wilson loop W, characterized by a closed current loop J:

$$Z[J] = \int Tr[W(J)] \exp[\frac{\beta}{2} \sum_{p_{\rho\nu}} Tr U_{\rho\nu}(p)] \mathcal{D}[U]. \tag{2.1}$$

The sum runs over all plaquettes $p_{\rho\nu}$ (with $\rho < \nu$), while $U_{\rho\nu}(p)$ is the product of the four $SU(2)$-matrices on the links of the plaquette. We will perform rather extensive manipulations in the following, therefore we fix our notation here:

$$\begin{aligned}
&p, q, r \quad \text{denote lattice points,} \\
&\mu, \nu, \rho, \lambda = 1, \cdots, d \quad \text{denote space directions,} \\
&p \pm \mu \quad \text{is the lattice point next to } p \quad \text{in positive or negative } \mu\text{-direction,} \\
&p_\mu \quad \text{denotes the link connecting } p \text{ with } p + \mu, \\
&p_{\mu\nu} \quad \text{with} \quad \mu < \nu \quad \text{is the plaquette determined by the links } p_\mu \text{ and } p_\nu, \\
&\Theta^a_\mu(p) = (\psi, \vartheta, \varphi)_\mu(p) \quad \text{denotes three angles which parametrize the SU(2) matrices} \\
&\text{on the link } p_\mu, \text{ indices } a, b \text{ generally run over } \psi, \vartheta, \varphi, \\
&U_\mu(p) \equiv U(\Theta^a_\mu(p)) \text{ is the } SU(2) \text{ matrix on the link } p_\mu, \\
&U_{\rho\nu}(p) = U_\rho(p) U_\nu(p + \rho) U^+_\rho(p + \nu) U^+_\nu(p) \text{ is the plaquette variable.}
\end{aligned} \tag{2.2}$$

As a first step we have to choose a parametrization for the link variables $U_\mu(p)$ in order to be able to do the group integrations. We proceed similarly as in previous work which applied the optimized δ-expansion on the lattice [3], [4]. In our case the Euler parametrization is the most appropriate, i.e. we use

$$U = e^{i\sigma_3\psi} e^{i\sigma_2\vartheta} e^{i\sigma_3\varphi}. \tag{2.3}$$

The Haar measure is proportional to $\sin(2\vartheta)$. A possible choice of the parameter space is $-\pi < \psi < \pi, \quad 0 < \varphi < \pi, \quad 0 < \vartheta < \pi/2$.

For the following it is convenient to extend this region. All integrals which appear contain functions of the $U_{\rho\nu}(p)$ which are periodic in the Euler angels. So we may use some symmetry relations which are easily seen from the decomposition of U into 1 and the σ_m. The shift $\varphi \to \varphi - \pi, \psi \to \psi - \pi$ leaves U invariant. Therefore one can extend the φ-integration into the interval from $-\pi$ to π, thus covering the group manifold twice. The further symmetry $\vartheta \to -\vartheta, \psi \to \psi - \pi/2, \varphi \to \varphi + \pi/2$ allows to extend the ϑ-integration to the interval $-\pi/2 < \vartheta < \pi/2$ if we continue the Haar measure as even function. Finally the symmetry $\vartheta \to \vartheta - \pi, \psi \to \psi - \pi$ allows the extension of the ϑ-integration to the full interval. Therefore we take

$$H(\vartheta) = \frac{\pi}{2} |\sin(2\vartheta)| \tag{2.4}$$

as Haar measure in the following, and use the common boundaries $-\pi < \psi, \vartheta, \varphi < \pi$.

In fig. 1 we show the notation for the link variables of the plaquette $p_{\rho\nu}$.

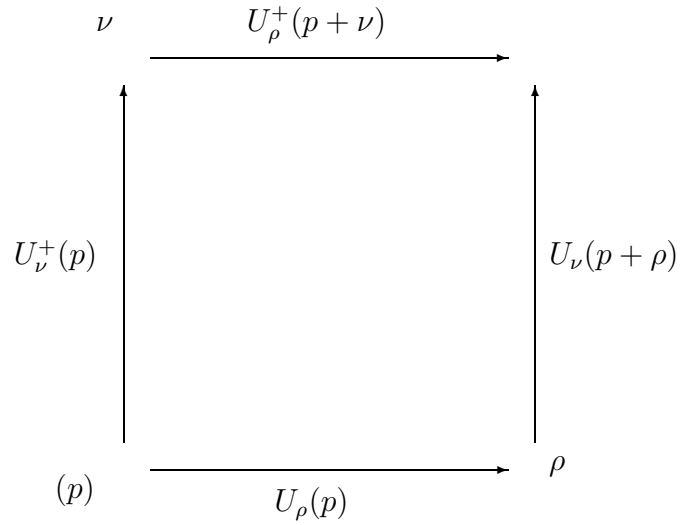

Fig.1. The plaquette $p_{\rho\nu}$ and the link variables.

The parametrization according to (2.3) becomes

$$\begin{aligned} U_\rho(p) &= e^{i\sigma_3\psi_\rho(p)}\, e^{i\sigma_2\vartheta_\rho(p)}\, e^{i\sigma_3\varphi_\rho(p)}, \\ U_\nu(p+\rho) &= e^{i\sigma_3\psi_\nu(p+\rho)}\, e^{i\sigma_2\vartheta_\nu(p+\rho)}\, e^{i\sigma_3\varphi_\nu(p+\rho)}, \\ U_\rho^+(p+\nu) &= e^{-i\sigma_3\varphi_\rho(p+\nu)}\, e^{-i\sigma_2\vartheta_\rho(p+\nu)}\, e^{-i\sigma_3\psi_\rho(p+\nu)}, \\ U_\nu^+(p) &= e^{-i\sigma_3\varphi_\nu(p)}\, e^{-i\sigma_2\vartheta_\nu(p)}\, e^{-i\sigma_3\psi_\nu(p)}. \end{aligned} \tag{2.5}$$

An appropriate technique for the further procedure, which was also extensively used in [3] and [4], is the splitting of the matrix exponentials into sums of ordinary exponentials times projection operators, in general

$$e^{\pm i\sigma_m\alpha} = \sum_{s=\pm1} e^{\pm is\alpha} P_s(m), \quad \text{with} \quad P_s(m) = \frac{1}{2}(1+s\sigma_m). \tag{2.6}$$

The link variables U are parametrized as in (2.5), the plaquette variable $U_{\rho\nu}(p)$ therefore contains 12 factors. To every factor we apply the decomposition (2.6). This means that we need 12 summation indices $s[\Theta] = \pm1$, associated with the twelve angles in (2.5). At the end $TrU_{\rho\nu}(p)$ becomes a product of 12 factors, each being a sum of two terms of the type (2.6). So in total we have a sum of 2^{12} terms. Each term of the sum consists of a trace T of a product of 12 projectors $P_s(m)$, multiplied by an exponential.

At the four corners of the plaquette one has a product of two projectors $P_s(3)P_{s'}(3)$. They both project on the same subspace, therefore we get a non-vanishing result only if the neighboring parameters s and s' agree. So we must have $s[\varphi_\rho(p)] = s[\psi_\nu(p+\rho)]$, $s[\varphi_\nu(p+\rho)] = s[\varphi_\rho(p+\nu)], s[\psi_\rho(p+\nu)] = s[\varphi_\nu(p)], s[\psi_\nu(p)] = s[\psi_\rho(p)]$, there are in fact not 12, but only 8 independent parameters.

We enumerate the remaining 8 independent parameters $s[\Theta]$ in a consecutive way, starting with $s[\vartheta_\rho(p)]$. In (2.7) we give the parameters s_i, together with the angles to which they refer. According to the remarks above, the s_i with even i belong to two angles.

$$
\begin{array}{lcll}
s_1 & \Leftrightarrow & \vartheta_\rho(p), & s_2 \Leftrightarrow \varphi_\rho(p), \psi_\nu(p+\rho), \\
s_3 & \Leftrightarrow & \vartheta_\nu(p+\rho), & s_4 \Leftrightarrow \varphi_\nu(p+\rho), \varphi_\rho(p+\nu), \\
s_5 & \Leftrightarrow & \vartheta_\rho(p+\nu), & s_6 \Leftrightarrow \psi_\rho(p+\nu), \varphi_\nu(p), \\
s_7 & \Leftrightarrow & \vartheta_\nu(p), & s_8 \Leftrightarrow \psi_\nu(p), \psi_\rho(p).
\end{array} \tag{2.7}
$$

The number of terms in the expansion of $TrU_{\rho\nu}(p)$ has now been reduced from 2^{12} to 2^8. We enumerate these terms by an index n, the order in which this is done needs not to be specified yet. Thus each s_i becomes a function of n, we denote it by s_{in}. The traces also depend upon n, we denote them by T_n. Finally one can write $TrU_{\rho\nu}(p)$ in the following form:

$$
TrU_{\rho\nu}(p) = Tr[U_\rho(p)U_\nu(p+\rho)U_\rho^+(p+\nu)U_\nu^+(p)] = \sum_{n=1}^{2^8} T_n \exp[iA^n_{\rho\nu}(p;\Theta,s)]. \tag{2.8}
$$

Here

$$
\begin{array}{rcl}
A^n_{\rho\nu}(p;\Theta,s) & = & s_{8n}\psi_\rho(p) + s_{1n}\vartheta_\rho(p) + s_{2n}\varphi_\rho(p) \\
& & +s_{2n}\psi_\nu(p+\rho) + s_{3n}\vartheta_\nu(p+\rho) + s_{4n}\varphi_\nu(p+\rho) \\
& & -s_{4n}\varphi_\rho(p+\nu) - s_{5n}\vartheta_\rho(p+\nu) - s_{6n}\psi_\rho(p+\nu) \\
& & -s_{6n}\varphi_\nu(p) - s_{7n}\vartheta_\nu(p) - s_{8n}\psi_\nu(p).
\end{array} \tag{2.9}
$$

The computation of the non-vanishing 2^8 traces T_n shows that they all have the values $\pm 1/16$. This structure is easily understood. Applying the projectors $P_s(3)$ and $P_s(2)$ to a given vector successively leads to alternating projections on different subspaces. Each projection gives a factor $\pm 1/\sqrt{2}$, with the sign depending upon the s_{in}. There are 144 values of n with $T_n = 1/16$ and 112 with $T_n = -1/16$. Obviously $\sum_n T_n = 2$ as it should. A further simplification is obtained from the symmetry $T_n(s_{in}) = T_n(-s_{in})$, which is easily seen by inserting $\sigma_1\sigma_1$ between all projectors and using $\sigma_1 P_s(m)\sigma_1 = P_{-s}(m)$ for $m = 2, 3$. Together with the symmetry $A^n_{\rho\nu}(p;\Theta,s_{in}) = -A^n_{\rho\nu}(p;\Theta,-s_{in})$ and the reality of the trace in (2.8) we can therefore fix, e.g. $s_{8n} = 1$, multiply (2.8) by 2 on the rhs, and restrict the sum over n to the 2^7 values $s_{1n}, \cdots, s_{7n}$. It is convenient to introduce sign factors $\epsilon_n = \pm 1$ and write

$$
T_n = \frac{\epsilon_n}{16} = Tr[(P_{s_{1n}}(2)P_{s_{2n}}(3))\ (P_{s_{3n}}(2)P_{s_{4n}}(3))\ (P_{s_{5n}}(2)P_{s_{6n}}(3))\ (P_{s_{7n}}(2)P_{s_{8n}=1}(3))]. \tag{2.10}
$$

According to the previous remarks we dropped redundant projectors and fixed $s_{8n} = 1$. We may now write (2.8) as sum over the restricted set n in the form

$$
TrU_{\rho\nu}(p) = \frac{1}{8}\sum_{n=1}^{2^7} \epsilon_n \cos[A^n_{\rho\nu}(p;\Theta,s)]. \tag{2.11}
$$

In order to perform the integrations over all the angles ψ, ϑ, φ we proceed essentially as in [1]. The various exponentials are expanded into a series of modified Bessel functions according to the formula

$$\exp[\epsilon z \cos A] = \sum_{l=-\infty}^{\infty} \epsilon^l I_l(z) e^{ilA} \quad \text{for } \epsilon = \pm 1. \tag{2.12}$$

The replacement $\epsilon \to -\epsilon$ is obviously equivalent to the shift $A \to A + \pi$. Using (2.11) and (2.12) we thus find

$$\begin{aligned}
\exp[\frac{\beta}{2} \sum_{p_{\rho\nu}} TrU_{\rho\nu}(p)] &= \exp\left[\frac{\beta}{16} \sum_{p_{\rho\nu},n} \epsilon_n \cos[A^n_{\rho\nu}(p;\Theta,s)]\right] \\
&= \prod_{p_{\rho\nu},n} \exp\left[\frac{\beta}{16} \epsilon_n \cos[A^n_{\rho\nu}(p;\Theta,s)]\right] \\
&= \prod_{p_{\rho\nu},n} \sum_{l=-\infty}^{\infty} (\epsilon_n)^l I_l(\frac{\beta}{16}) \exp[ilA^n_{\rho\nu}(p;\Theta,s)] \\
&= \sum_{l^n_{\rho\nu}(p)} \prod_{p_{\rho\nu},n} (\epsilon_n)^{l^n_{\rho\nu}(p)} I_{l^n_{\rho\nu}(p)}(\frac{\beta}{16}) \exp[il^n_{\rho\nu}(p) A^n_{\rho\nu}(p;\Theta,s)].
\end{aligned} \tag{2.13}$$

In the last step we exchanged the order of the product and the sum. The summation parameters l must then be distinguished by indices referring to the corresponding plaquette $p_{\rho\nu}$ and the index n; the $l^n_{\rho\nu}(p)$ run independently from $-\infty$ to ∞.

We can now write down the expectation value for a Wilson loop. We characterize it by a closed current loop $J_\lambda(q)$ which is ± 1 if the current runs in or against the direction of the link q_λ, and 0 otherwise. For simplicity we exclude loops which run multiply through some links. For the calculation of the trace $Tr[W(J)]$ we use again the decomposition (2.6) for the link variables on the loop, parametrized by (2.3). For every link q_λ on the loop we have two parameters $\hat{s}^b_\lambda(q) = \pm 1$ (not three, because again neighboring projectors have to coincide), the whole set of these will be denoted by $\hat{s}$. For a loop of lengt L we have $2L$ parameters $\hat{s}^b_\lambda(q) = \pm 1$ and in total a sum of 2^{2L} terms. We count these by an index $\hat{n}$, and denote the parameters by $s^b_{\lambda\hat{n}}(q)$. The traces are called $W_{\hat{n}}$. The $W_{\hat{n}}$ have a similar structure as the T_n; their values are $\pm 1/2^L$, and $\sum_{\hat{n}} W_{\hat{n}} = 2$. The partition function now reads

$$\begin{aligned}
Z[J] &= \int \sum_{\hat{n}} W_{\hat{n}} \exp[i \sum_{q\lambda b} J_\lambda(q) \hat{s}^b_{\lambda\hat{n}}(q) \Theta^b_\lambda(q)] \\
&\quad \sum_{l^n_{\rho\nu}(p)} \left(\prod_{p_{\rho\nu},n} (\epsilon_n)^{l^n_{\rho\nu}(p)} I_{l^n_{\rho\nu}(p)}(\frac{\beta}{16}) \exp[il^n_{\rho\nu}(p) A^n_{\rho\nu}(p;\Theta,s)] \right) \\
&\quad \prod_{r\mu} \left(H(\vartheta_\mu(r)) \prod_a \frac{d\Theta^a_\mu(r)}{2\pi} \right).
\end{aligned} \tag{2.14}$$

Obviously the first sum over $\hat{n}$ is the expansion of the loop. The sum over $l^n_{\rho\nu}(p)$ in the second line contains the action transformed as in (2.13), while the product over r, μ

in the third line contains the integrations together with the Haar measure. We may factorize the product over $p_{\rho\nu}, n$ and apply the addition theorem for the exponentials. This results in

$$\begin{aligned} Z[J] &= \int \sum_{\hat{n}} W_{\hat{n}} \exp[i \sum_{q\lambda b} J_\lambda(q) \hat{s}^b_{\lambda\hat{n}}(q) \Theta^b_\lambda(q)] \\ &\sum_{l^n_{\rho\nu}(p)} \left(\prod_{p_{\rho\nu},n} (\epsilon_n)^{l^n_{\rho\nu}(p)} I_{l^n_{\rho\nu}(p)}(\frac{\beta}{16}) \right) \exp[i \sum_{p,\rho<\nu,n} l^n_{\rho\nu}(p) A^n_{\rho\nu}(p;\Theta,s)] \\ &\prod_{r\mu} \left(H(\vartheta_\mu(r)) \prod_a \frac{d\Theta^a_\mu(r)}{2\pi} \right). \end{aligned} \tag{2.15}$$

We are now ready to perform the angular integrations over $\Theta^a_\mu(r)$, remembering the definition of $A^n_{\rho\nu}(p;\Theta,s)$ in (2.9). The integrations over $\psi_\mu(r)$ and $\varphi_\mu(r)$ lead to a Kronecker-δ which gives a constraint, while the $\vartheta_\mu(r)$ integration involves the Haar measure (2.4) and leads to a more complicated function. We shall also call it a constraint for simplicity. It is convenient to define a symbol $\delta^a(C)$ for integer C by

$$\delta^a(C) = \left\{ \begin{array}{l} \int_{-\pi}^{\pi} e^{iC\psi} \frac{d\psi}{2\pi} = \delta_{C,0} \text{ for } a = \psi, \varphi, \\ \int_{-\pi}^{\pi} H(\vartheta) e^{iC\vartheta} \frac{d\vartheta}{2\pi} = \left\{ \begin{array}{ll} 1/(1-C^2/4) & \text{if } C \text{ is a multiple of 4,} \\ 0 & \text{otherwise} \end{array} \right\} \text{ for } a = \vartheta. \end{array} \right. \tag{2.16}$$

The argument of the constraint which arises from the $\Theta^a_\mu(r)$-integration becomes

$$\begin{aligned} C^a_\mu(r) &\equiv \sum_{\nu>\mu,n} [s_{(8,1,2)n} l^n_{\mu\nu}(r) - s_{(6,5,4)n} l^n_{\mu\nu}(r-\nu)] \\ &- \sum_{\nu<\mu,n} [s_{(8,7,6)n} l^n_{\nu\mu}(r) - s_{(2,3,4)n} l^n_{\nu\mu}(r-\nu)] + J_\mu(r) \hat{s}^a_{\mu\hat{n}}(r), \end{aligned} \tag{2.17}$$

where one has to use the first, second, or third subscript on s for $a = \psi, \vartheta, \varphi$ respectively. So we end up with the following expression for the expectation value of the loop:

$$Z[J] = \sum_{\hat{n}} W_{\hat{n}} \sum_{l^n_{\rho\nu}(p)} \left(\prod_{p_{\rho\nu},n} (\epsilon_n)^{l^n_{\rho\nu}(p)} I_{l^n_{\rho\nu}(p)}(\frac{\beta}{16}) \right) \prod_{r\mu a} \delta^a[C^a_\mu(r)]. \tag{2.18}$$

3 Integrating out unnecessary variables

The constraint equations (2.17), as they stand, have a quite different character than the corresponding ones in the abelian case in [1]. There one had one parameter $l_{\mu\nu}(r)$ for every plaquette $r_{\mu\nu}$, in our case we have 2^7 parameters $l^n_{\mu\nu}(r)$ which are characterized by the additional index n. Any attempt of a physical interpretation of the $l^n_{\mu\nu}(r)$ would be premature at this stage.

Let us consider a fixed plaquette $r_{\mu\nu}$ for the moment, and suppress the indices $r_{\mu\nu}$. We first look for a suitable linear transformation from the parameters l^n to new parameters

m^i. There are 8 combinations of the l^n which play a special role, namely the 8 sums $\sum_{n=1}^{128} s_{in} l^n$ which appear in the constraints (2.17) (remember that s_{8n} was fixed to 1). We will choose these 8 combinations as new variables m^i, $i = 1, \cdots, 8$, eliminate the first eight l^n, and keep the rest of the parameters as they are. The 8×8 matrix made up of the s_{ij} with $i, j = 1, \cdots, 8$ will be denoted by S.

One may consider the s_{in} as a set of 2^7 vectors $\mathbf{S}_n = (s_{1n}, \cdots, s_{7n}, s_{8n} = 1)$ with components ± 1. Up to now we did not specify the order in which these vectors $\mathbf{S}_n$ should be enumerated. It is now convenient to choose the $\mathbf{S}_n$ associated with the first eight values of n in such a way that $\epsilon_n = 1$ for $n = 1, \cdots, 8$, with ϵ_n the signs of the trace (2.10). This can be done in many ways. To be definite we give our choice in the appendix. Our criteria were a small but non-vanishing determinant of S (it is 128 for the S in A.1), and a structure as transparent as possible in some of the equations below. The transformation finally becomes

$$\begin{aligned} m^i &= \sum_{n=1}^{128} s_{in} l^n = \sum_{j=1}^{8} s_{ij} l^j + \sum_{\alpha=9}^{128} s_{i\alpha} l^\alpha \quad \text{for} \quad i = 1, \cdots, 8, \\ m^\alpha &= l^\alpha \quad \text{for} \quad \alpha = 9, \cdots, 128. \end{aligned} \tag{3.1}$$

In matrix form the transformation reads

$$\begin{pmatrix} m_c \\ m_f \end{pmatrix} = \begin{pmatrix} S & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l_c \\ l_f \end{pmatrix}, \quad \begin{pmatrix} l_c \\ l_f \end{pmatrix} = \begin{pmatrix} S^{-1} & -S^{-1}T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_c \\ m_f \end{pmatrix}. \tag{3.2}$$

We have split m into an eight dimensional "constrained" vector m_c, and a 120-dimensional "free" vector m_f. Here S is the 8×8 matrix defined before, with $S_{ij} = s_{ij}$. T is an 8×120 matrix, if we enumerate the columns from 9 to 128 we have $T_{i\alpha} = s_{i\alpha}$.

There is an important restriction which has to be imposed on the transformation. It is this restriction which will later on lead to the area law. If the l^n run over all integers, the same will be true for the m^α with $\alpha = 9, \cdots, 128$, as seen from (3.1). But it is not true for the m^i with $i = 1, \cdots, 8$. From the inversion of the transformation in (3.2) one finds that only those m^i appear, which fulfill the condition

$$l^j = \sum_{i=1}^{8} (S^{-1})_{ji} m^i - \sum_{i=1}^{8} (S^{-1})_{ji} \sum_{\alpha=9}^{128} s_{i\alpha} m^\alpha \stackrel{!}{=} \text{ integer for } j = 1, \cdots, 8. \tag{3.3}$$

A computation of the matrix elements $(S^{-1})_{ji}$ shows that they are integer or half integer, and that the sums over the elements of any row are integer. Because the $s_{i\alpha}$ are ± 1, this implies that the second sum in (3.3) is automatically integer. Therefore the condition simplifies to $\sum_{i=1}^{8} (S^{-1})_{ji} m^i \stackrel{!}{=}$ integer for $j = 1, \cdots, 8$. This restriction finally becomes equivalent to

$$m^i = \text{ even for all } i, \text{ or } m^i = \text{ odd for all } i = 1, \cdots, 8. \tag{3.4}$$

Introducing the transformation (3.2) into the product in (2.18) one obtains

$$\sum_{l^n}\prod_n(\epsilon_n)^{l^n}I_{l^n}(\frac{\beta}{16})=\sum_{m^i}\sum_{m^\alpha}\left(\prod_{j=1}^{8}I_{l^j(m^i,m^\alpha)}(\frac{\beta}{16})\right)\prod_{\alpha=9}^{128}(\epsilon_\alpha)^{m^\alpha}I_{m^\alpha}(\frac{\beta}{16}), \tag{3.5}$$

with $l^j(m^i,m^\alpha)=\sum_i(S^{-1})_{ji}m^i-\sum_\alpha(S^{-1}T)_{j\alpha}m^\alpha$. The sum over m^i only runs over the subset which fulfills (3.4). The m^α for $\alpha=9,\cdots,128$ do not show up in the constraints, therefore the summations can be performed. To do this we introduce the integral representation for all the modified Bessel functions, thereby partially reversing the previous step of expanding the exponents.

$$\epsilon^l I_l(\frac{\beta}{16})=\int_{-\pi}^{\pi}\frac{d\gamma}{2\pi}\exp[\epsilon\frac{\beta}{16}\cos\gamma+il\gamma]\quad\text{for }\epsilon=\pm1. \tag{3.6}$$

The m^α appear in the exponent now, and the summations can be performed with the help of the Poisson sum formula

$$\sum_{m^\alpha}\exp\left[i[\gamma_\alpha-\sum_{j=1}^{8}\gamma_j(S^{-1}T)_{j\alpha}]m^\alpha\right]=2\pi\sum_{k_\alpha}\delta[\gamma_\alpha-\sum_{j=1}^{8}\gamma_j(S^{-1}T)_{j\alpha}-2\pi k_\alpha]. \tag{3.7}$$

All γ_α are integrated over the interval $-\pi<\gamma_\alpha\leq\pi$, thus exactly one k_α contributes in the sum on the rhs. Furthermore the γ_α appear only as arguments of periodic cosines in the remaining part of the integrand. Therefore we may simply use the δ-functions to eliminate the γ_α by putting $\gamma_\alpha=\sum_{j=1}^{8}\gamma_j(S^{-1}T)_{j\alpha}$ for $\alpha=9,\cdots,128$. We thus are left with

$$\sum_{l^n}\prod_n(\epsilon_n)^{l^n}I_{l^n}(\frac{\beta}{16})=\sum_{m^i}\int\frac{d\gamma_1}{2\pi}\cdots\frac{d\gamma_8}{2\pi}\exp[\frac{\beta}{16}A(\gamma)+i\sum_{i,j=1}^{8}\gamma_j(S^{-1})_{ji}m^i], \tag{3.8}$$

where we have introduced the function

$$A(\gamma)=\sum_{j=1}^{8}\cos\gamma_j+\sum_{\alpha=9}^{128}\epsilon_\alpha\cos[\sum_{j=1}^{8}\gamma_j(S^{-1}T)_{j\alpha}]. \tag{3.9}$$

In order to fulfill the conditions (3.4) we put

$$m^i=2\tilde{m}^i+f,\text{ with }\tilde{m}^i\text{ integer, and }f=0\text{ or }1. \tag{3.10}$$

For our choice of the transformation, the f-dependence of (3.8) becomes particularly simple and only involves γ_8.

We now reintroduce the suppressed indices $p_{\rho\nu}$ into (3.8), (3.10), and insert the result into the partition function (2.18). We find

$$\begin{aligned}Z[J]=\quad&\sum_{\hat{n}}W_{\hat{n}}\sum_{\tilde{m}^i_{\rho\nu}(p)}\sum_{f_{\rho\nu}(p)}\prod_{p_{\rho\nu}}\left(\int\frac{d\gamma_1}{2\pi}\cdots\frac{d\gamma_8}{2\pi}\right.\\&\left.\exp\left[\frac{\beta}{16}A(\gamma)+2i\sum_{i,j=1}^{8}\gamma_j(S^{-1})_{ji}\tilde{m}^i_{\rho\nu}(p)+i\gamma_8 f_{\rho\nu}(p)\right]\right)\prod_{r\mu a}\delta^a[C^a_\mu(r;\tilde{m},f,J,\hat{s})].\end{aligned} \tag{3.11}$$

The constraints (2.17) now depend upon $\tilde{m}^i$ and f. For the formulation it is convenient to extend $f_{\mu\nu}(r)$ to an antisymmetric matrix. Thus $f_{\mu\nu}(r) = 0, 1$ for $\mu < \nu$, and $f_{\mu\nu}(r) = 0, -1$ for $\mu > \nu$. The constraints then become

$$\begin{aligned} C^a_\mu(r) &= 2\sum_{\nu>\mu}[\tilde{m}^{(8,1,2)}_{\mu\nu}(r) - \tilde{m}^{(6,5,4)}_{\mu\nu}(r-\nu)] - 2\sum_{\nu<\mu}[\tilde{m}^{(8,7,6)}_{\nu\mu}(r) - \tilde{m}^{(2,3,4)}_{\nu\mu}(r-\nu)] \\ &+ \sum_{\nu\neq\mu}\Delta_\nu f_{\mu\nu}(r) + J_\mu(r)\hat{s}^a_{\mu\hat{n}}(r), \end{aligned} \tag{3.12}$$

where Δ_ν denotes the left lattice derivative. The tensor $f_{\mu\nu}(r)$ will be recognized as the Z_2 field strength tensor. As in (2.17) one has to use the first, second, or third upper index of $\tilde{m}_{\mu\nu}$ for $a = \psi, \vartheta, \varphi$.

There is a symmetry relation in (3.11) which underlines the importance of the Z_2 tensor $f_{\rho\nu}(p)$. Replace $\beta \to -\beta$ and substitute $\gamma_j \to \gamma_j + \pi$ for all j (the integrand is periodic). Using the definition of $A(\gamma)$ in (3.9) and the fact that $\sum_{j=1}^8 (S^{-1}T)_{j\alpha} = 1$ for all α, one finds that $A(\gamma)$ reverses sign, i.e. $\beta A(\gamma)$ stays invariant. The second term in the exponent of (3.11) changes by a multiple of $2\pi i$, because $\sum_{j=1}^8 (S^{-1})_{ji} = \delta_{8i}$, and $\tilde{m}^8_{\rho\nu}(p)$ is integer. Finally the third term changes by $i\pi f_{\rho\nu}(p)$. In this way one finds that the bracket $(\cdots)$ in (3.11) is even in β for $f_{\rho\nu}(p) = 0$, and odd in β for $f_{\rho\nu}(p) = 1$.

Up to this point all formulae were exact. It appears tempting now to proceed as follows in the continuum limit of large β. If the function $A(\gamma)$ has an isolated maximum, the integrals over γ_j are dominated by the region where $A(\gamma)$ becomes maximal, and the integrations over the γ_j can be extended to the interval from $-\infty$ to ∞. A quadratic expansion around the maximum would then lead to gaussian integrals. If all the ϵ_α were equal to 1, we would indeed have a simple maximum at $\gamma_j = 0$ for all γ_j. This would correspond to the situation in the abelian theory and to the replacement $I_l(z) \to e^z e^{-l^2/2z}/\sqrt{2\pi z}$ which was used in [1] (these authors used by mistake $e^{-l^2/4z}$ instead of the correct $e^{-l^2/2z}$, which has, however, no consequences there).

In our case the different signs of the ϵ_α change the situation drastically. One finds that the function $A(\gamma)$ assumes its maximal value of 16 if $\gamma_1, \gamma_2, \gamma_3$ are arbitrary, and $\gamma_j = 0$ for $j = 4, \cdots, 8$. Even a quadratic approximation in γ_j for $j = 4, \cdots, 8$, and fixed $\gamma_1, \gamma_2, \gamma_3$ is not possible because the matrix of the second derivatives has two zero eigenvalues. For $\gamma_1{=}\gamma_2{=}\gamma_3{=}0$ even three eigenvalues vanish. So it would be necessary to go to a higher order in the expansion; but then the integrations are no longer gaussian and cannot be performend. This is the way in which non abelian gauge theory protects itself from being solved analytically!

Nevertheless the present formalism will clearly show, how the area law for large loops arises. We keep β arbitrary, not necessarily large, and first solve the constraints.

4 Solution of the constraints

It is convenient to rewrite the ϑ-constraints in the form

$$\delta^\vartheta[C^\vartheta_\mu(r)] = \sum_{k_\mu(r)=-\infty}^{\infty} \delta^\vartheta[4k_\mu(r)]\delta_{C^\vartheta_\mu(r)-4k_\mu(r),0}. \tag{4.1}$$

This introduces additional sums over the $k_\mu(r)$, and factors $\delta^\vartheta[4k_\mu(r)]$. The ϑ-constraints now also appear in form of a Kronecker-δ.

Let us first consider the constraints modulo 2, which obviously only concerns the second line of (3.12). The factors $\hat{s}^a_{\mu\hat{n}}(r) = \pm 1$ may be dropped, and for all three cases $a = \psi, \vartheta, \varphi$ we find the equations

$$\sum_\nu \Delta_\nu f_{\mu\nu}(r) + J_\mu(r) = 0 \pmod 2 \text{ for all } r, \mu. \tag{4.2}$$

These are identical to the equations for $l_{\mu\nu}(r)$ in the abelian case, except that they are equations modulo 2. They show already the appearance of a Z_2 structure. We concentrate on a geometrical formulation of the solution. Recall that $f_{\mu\nu}(r) = 0, 1$ for $\mu < \nu$ and that $f_{\mu\nu}(r)$ is antisymmetric. Therefore it is convenient to use the symbol $\epsilon_{\mu\nu} = (1, -1, 0)$ for $(\mu < \nu, \mu > \nu, \mu = \nu)$. Let S be a two dimensional surface, and

$$f^{(S)}_{\mu\nu}(r) = \begin{cases} \epsilon_{\mu\nu} & \text{if the plaquette } r_{\mu\nu} \text{ is part of the surface } S, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

The following statements hold:

- If S has the Wilson loop as boundary, then $f^{(S)}_{\mu\nu}(r)$ is a solution of (4.2).
- If S is a closed surface, then $f^{(S)}_{\mu\nu}(r)$ is a solution of the homogeneous equation $\sum_\nu \Delta_\nu f^{(H)}_{\mu\nu}(r) = 0 \pmod 2$. The most general solution can be obtained as a superposition of a special solution $f^{(S)}_{\mu\nu}(r)$ with S bounded by the loop, and a sum over solutions of the homogeneous equation.

The proof is obvious. Equation (4.2) involves exactly all the plaquettes which contain the link r_μ. Links r_μ on the loop appear in an odd number of plaquettes of the associated surface S, while links r_μ which are not part of the loop appear in an even number (including 0) of plaquettes of S.

In the following we will restrict the discussion to planar loops for simplicity. A solution of special importance is the layer belonging to the minimal surface of the Wilson loop W,

$$f^{(min)}_{\mu\nu}(r) = \begin{cases} \epsilon_{\mu\nu} & \text{if the plaquette } r_{\mu\nu} \text{ is part of the minimal surface,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

The $f^{(min)}_{\mu\nu}(r)$ associated with the minimal surface fulfills (4.2) exactly, not only modulo 2, if the loop is oriented appropriately. For different surfaces, on the other hand, this is not true.

The general solution of the homogeneous equation $\sum_\nu \Delta_\nu f^{(H)}_{\mu\nu}(r) = 0 \pmod 2$ can be written down explicitly. It depends upon the dimension d. In order to guarantee the antisymmetry of $f_{\mu\nu}(r)$ we introduce the symbol $(\mathrm{mod}_{\mu\nu}\ 2)$; it is identical with (mod 2) for $\mu < \nu$, but reverses sign for $\mu > \nu$. One then has

$$f^{(H)}_{\mu\nu}(r) = \begin{cases} \sum_\lambda \epsilon_{\mu\nu\lambda} \Delta_\lambda f(r) & (\mathrm{mod}_{\mu\nu}\ 2) \text{ for } d = 3, \\ \sum_{\lambda\kappa} \epsilon_{\mu\nu\lambda\kappa} \Delta_\lambda f_\kappa(r) & (\mathrm{mod}_{\mu\nu}\ 2) \text{ for } d = 4. \end{cases} \tag{4.5}$$

The function $f(r)$ in three dimensions is unique up to a constant. For $d = 4$ one has a gauge freedom, i.e. adding a gradient $\Delta_\kappa \Lambda(r)$ to $f_\kappa(r)$ will not change $f^{(H)}_{\mu\nu}(r)$. The simplest way to remove this ambiguity is to choose an axial gauge by imposing $\sum_\kappa n_\kappa f_\kappa(r) = 0 \pmod 2$. In both cases the values of $f(r)$ and $f_\kappa(r)$, respectively, are restricted to 0 and 1.

Switching from one surface S to another S' for the special solution can also be rephrased in terms of the solution of the homogeneous equation. In $d = 3$ dimensions it corresponds to changing $f(r)$ by 1 inside the volume between the two surfaces. (The use of the left derivative in (4.2), (4.5) specifies which points of the surface have to be considered as inside or outside). For $d = 4$ one has to choose a three-dimensional volume spanned by the surfaces with, roughly speaking, normal vector in κ-direction at the point r. One then has to change $f_\kappa(r)$ by 1 inside the volume, and subsequently transform to the axial gauge.

The essential part of the constraints has now been solved. We put

$$f_{\mu\nu}(r) = f^{(min)}_{\mu\nu}(r) + \sum_{\lambda\kappa} \epsilon_{\mu\nu\lambda\kappa} \Delta_\lambda f_\kappa(r) \quad (\mathrm{mod}_{\mu\nu}\ 2). \tag{4.6}$$

The minimal surface layer $f^{(min)}_{\mu\nu}(r)$, defined in (4.4), is no longer a variable, but uniquely fixed by the loop. The $f_\kappa(r) = 0, 1$ are unconstrained. The index κ on $\epsilon_{\mu\nu\lambda\kappa}$, $f_\kappa(r)$ and in the sum appears for $d = 4$ only. For $d = 3$ it has to be dropped, here and wherever it appears in subsequent formulae.

We next introduce the solution (4.6) into (3.12). The second line is now definitely even, therefore we denote it by

$$2R^a_{\mu\hat{n}}(r) \equiv \sum_\nu \Delta_\nu f_{\mu\nu}(r) + J_\mu(r) \hat{s}^a_{\mu\hat{n}}(r). \tag{4.7}$$

For later use we specify the variables upon which $R^a_{\mu\hat{n}}(r)$ can depend. The term $J_\mu(r)\hat{s}^a_{\mu\hat{n}}(r)$ is strictly local, i.e. only depends on the argument r. The $f_\kappa(r)$, on the other hand, appear as $\Delta_\nu \Delta_\lambda f_\kappa(r)$ with $\nu \neq \lambda$. Therefore they enter also with shifted arguments r'. The points r' and r are neighbors in the sense that all components of $r - r'$ are either 0 or 1. Finally, one has to note that (4.6) is only an equation modulo 2. Therefore $\sum_\nu \Delta_\nu f_{\mu\nu}(r)$ as well as $R^a_{\mu\hat{n}}(r)$ can also depend on the minimal layer $f^{(min)}_{\mu\nu}(r)$.

The constraints (3.12) now become

$$\begin{aligned} C^a_\mu(r) = \quad & 2\sum_{\nu>\mu} [\tilde{m}^{(8,1,2)}_{\mu\nu}(r) - \tilde{m}^{(6,5,4)}_{\mu\nu}(r-\nu)] - 2\sum_{\nu<\mu} [\tilde{m}^{(8,7,6)}_{\nu\mu}(r) - \tilde{m}^{(2,3,4)}_{\nu\mu}(r-\nu)] \\ & + 2R^a_{\mu\hat{n}}(r) \stackrel{!}{=} 4\delta^{a\vartheta} k_\mu(r). \end{aligned} \tag{4.8}$$

For $a = \psi$ (first upper index i on $\tilde{m}^i$) and $a = \varphi$ (third upper index i on $\tilde{m}^i$), i.e. for the even indices i, the index i appears in both sums of (4.8). For a fixed r one has $2d$ linear equations (corresponding to $a = \psi, \varphi$ and $\mu = 1, \cdots, d$) for $4d(d-1)/2$ quantities $\tilde{m}^{(2,4,6,8)}_{\mu\nu}$. These equations are not independent due to the identity

$$\sum_\mu [C^\psi_\mu(r) - C^\varphi_\mu(r-\mu)] = 2\sum_\mu [R^\psi_{\mu\hat{n}}(r) - R^\varphi_{\mu\hat{n}}(r-\mu)]. \tag{4.9}$$

The rhs of (4.9) vanishes because $f_{\mu\nu}$ is antisymmetric, and because the neighboring projectors in the loop have to coincide as mentioned in sect. 2. We checked explicitly for $d = 3, 4$ that the equations may be simply used to eliminate some of the $\tilde{m}^i_{\mu\nu}(r)$. Any eliminated $\tilde{m}^i_{\mu\nu}(r)$ depends linearly on other unconstrained $\tilde{m}^{i'}_{\mu'\nu'}(r')$ and on $R^a_{\mu'\hat{n}}(r')$, where the components of r'-r are either 0 or ± 1.

For $a = \vartheta$ (second upper index i on $\tilde{m}^i$), i.e. for the odd indices i, the situation is even simpler. Each index i enters only in one of the sums in (4.8), it is convenient to eliminate some of the $\tilde{m}^1_{\mu\nu}(r)$ and $\tilde{m}^7_{\nu\mu}(r)$.

Finally we can write the solutions of the constraints in the following form, which eliminates some of the $\tilde{m}^i_{\mu\nu}(r)$, leaving the rest unconstrained.

$$\tilde{m}^i_{\mu\nu}(r) = L^i_{\mu\nu}[\tilde{m}^{i'}_{\mu'\nu'}(r'), R^a_{\mu'\hat{n}}(r')] + 2k^i_{\mu\nu}(r), \tag{4.10}$$

with $k^i_{\mu\nu}(r) = (k_\mu(r), k_\nu(r), 0)$ for $(i = 1, i = 7$,otherwise). The $L^i_{\mu\nu}$ are linear combinations of their arguments, only coefficients $0, \pm 1$ appear. The arguments r, r' are neighbors in the sense explained before. The loop current $J_\mu(r)$ enters only in $R^a_{\mu'\hat{n}}(r')$.

Note the drastic difference in the type of the constraints (4.2) (or the corresponding constraints in eq. (6) of ref [1] for the $U(1)$ case) on one hand, and the constraints (4.8) just considered on the other. The former involve a difference operator applied to one plaquette variable $f_{\mu\nu}(r)$, the corresponding Green function being non-local and coupling the solution to the current over a long range. In contrast, the latter constraints involve several plaquette variables $\tilde{m}^i_{\mu\nu}(r)$ and can just be used to eliminate some of these. This elimination leads to an almost local coupling to the current, involving neighbors only.

5 Confinement

The essential feature, which finally arose in our formulation, is the presence of the Z_2 field strength tensor $f_{\mu\nu}(r)$ which obeys the field equation (4.2). The solutions of this equation can be characterized by two-dimensional surfaces; a layer with the Wilson loop as boundary, possibly together with closed surfaces. One may expect that the presence of such a layer will lead to an area law. For a qualitative understanding of the confinement mechanism we use the explicit form (4.6) for the solution of the field equation (4.2). It contains the fixed layer $f^{(min)}_{\mu\nu}(r)$, together with the unconstrained Z_2 variable $f_\kappa(r)$. The solutions of the remaining constraint equations (4.8) for the $\tilde{m}^i_{\mu\nu}(r)$ have the form (4.10).

Consider now the expression (3.11) for the partition function $Z[J]$, and introduce the solutions (4.6), (4.10) of the constraints into the exponential on the rhs. This gives

$$\begin{aligned} & 2\sum_{i,j=1}^{8} \gamma_j (S^{-1})_{ji} \tilde{m}^i_{\rho\nu}(p) + \gamma_8 f_{\rho\nu}(p) \\ &= 2\sum_{i,j=1}^{8} \gamma_j (S^{-1})_{ji} \{L^i_{\rho\nu}[\tilde{m}^{i'}_{\rho'\nu'}(p'), R^a_{\rho'\hat{n}}(p')] + 2k^i_{\rho\nu}(p)\} \\ &+ \gamma_8 \{f^{(min)}_{\rho\nu}(p) + \sum \epsilon_{\rho\nu\lambda\kappa} \Delta_\lambda f_\kappa(p) \pmod{\mu\nu\ 2}\}. \end{aligned} \tag{5.1}$$

The Wilson loop enters into this expression in two different ways. First there is a dependence on the current $J_{\rho'}(p')\hat{s}^a_{\rho'\hat{n}}(p')$ which arises from the second term of $R^a_{\rho'\hat{n}}(p')$ in (4.7). Secondly there is a dependence on the minimal layer $f^{(min)}_{\rho\nu}(p)$ which enters into the first term of $R^a_{\rho'\hat{n}}(p')$, as well as explicitely in the factor of γ_8.

The current J is present on a one-dimensional set, the minimal layer $f^{(min)}$ on a two-dimensional set. Besides this, both quantities enter in a quite similar way into (5.1). One may therefore expect, that for large loops the dependence on the one-dimensional current J can be neglected compared to the dependence on the two-dimensional layer $f^{(min)}$.

If we neglect the dependence on $J_{\rho'}(p')$ the partition function becomes independent of the $\hat{s}^a_{\rho'\hat{n}}(p')$ and the sum $\sum_{\hat{n}} W_{\hat{n}} = 2$ can be performed. Assuming that the γ-integrations have been done, the degrees of freedom are now in the remaining unconstrained $\tilde{m}^i_{\rho\nu}(p) = -\infty, \cdots, \infty$, the $f_\kappa(p) = 0, 1$, and the $k_\mu(r) = -\infty, \cdots, \infty$ introduced at the beginning of sect. 4.

The discussion of (4.8) showed that the solutions couple neighbors only. This means that (5.1), which appears in the exponential in (3.11), only depends on these variables with arguments p, p', p''; here p', p'' are neighbors in the sense that all components of p'-p are $0, \pm 1$, all components of p''-p are $0, \pm 1, \pm 2$.

We digress for a technical point. Neither the exponential in (3.11) with it's complex argument, nor the factors $\delta^\vartheta[4k]$ are positive definite. Actually, according to the definition (2.16), one has $\sum_k \delta^\vartheta[4k] = 0$, because the Haar measure fulfills $H(0) = 0$. If desired, one could bring the expression into the usual form of a partition function with positive summands, by performing a twofold partial summation with respect to the $k_\mu(r)$.

The whole loop dependence is now in the $f^{(min)}_{\rho\nu}(p)$ belonging to the minimal surface. It acts like a space-time dependent external field, comparable, say, to a constant magnetic field switched on in a finite volume of an Ising model. $Z[J]$ is a partition function where the variables couple to neighbors only, a well known standard situation in statistical mechanics. For large subsystems it therefore factorizes into products refering to the subsystems and, correspondingly, has an exponential dependence on the volume. This is, of course, nothing else but the fact that the free energy is an extensive quantity. Rigorous proofs, which apply for any dimension, can be found in [5].

Consider now a loop $0 < x_1 \leq R, 0 < x_4 \leq T$ in the x_1-x_4-plane for definiteness, with R and T large. We divide the x_1-x_4-plane inside, as well as outside of the loop, into rectangles; these rectangles are then extended to d-dimensional boxes into the orthogonal directions. This means that we define regions $V^{(n)}$ by the inequalities $r_1^{(n)} < x_1 \leq r_2^{(n)}, t_1^{(n)} < x_4 \leq t_2^{(n)}, x_2, x_3$ arbitrary. The rectangle $r_1^{(n)} < x_1 \leq r_2^{(n)}, t_1^{(n)} < x_4 \leq t_2^{(n)}$ has to lie either completely inside the loop, or completely outside the loop. If not only R and T, but also all the differences $r_2^{(n)} - r_1^{(n)}$ and $t_2^{(n)} - t_1^{(n)}$ are large, the partition function will factorize,

$$Z[J] = \prod_n Z^{(n)}. \tag{5.2}$$

Consider now the ratio $Z[J]/Z[0]$, with $Z[0]$ the expression without loop. Obviously all the outer factors cancel. For the inner ones, on the other hand, one has $f^{(min)}_{\rho\nu}(p) = 1$

in the numerators, but 0 in the denominators. Thus the ratios are different from 1. Because of the factorization property, the volumes of the regions $V^{(n)}$ have to enter in the exponent. This finally implies that the area $A = R \times T$ of the loop enters in the exponent, so the result may be written as

$$Z[J]/Z[0] = \exp[-\sigma A]. \tag{5.3}$$

We have obtained the area law for large loops.

Several comments are appropriate here.

First one may wonder what would happen with our argumentation, if we would replace $f^{(min)}_{\rho\nu}(p)$, associated with the minimal surface, by a solution $f^{(S)}_{\rho\nu}(p)$, belonging to a different surface S. Obviously the simplicity of the situation for the regions $V^{(n)}$ inside and outside would break down, and factorization would not lead to a simple relation. The minimal surface is really unique for the argumentation.

The neglection of the dependence on J would certainly have been wrong if performed in the original expression described by the Euler angles $\psi_\mu(r), \vartheta_\mu(r)$, $\varphi_\mu(r)$. There one had only the current J but no layer $f^{(min)}$ showed up. Therefore confinement has to arise from the dependence on J in a complicated way. In our formulation the formalism led to another quantity, the minimal layer $f^{(min)}$. This appears as the natural quantity which describes the long distance physics and dominates the residual direct dependence on J.

For illustration one can have a look on the strong coupling limit. According to the discussion at the end of sect. 3, the bracket $(\cdots)$ in (3.11) is even in β for $f_{\rho\nu}(p) = 0$, and odd in β for $f_{\rho\nu}(p) = \epsilon_{\rho\nu}$. Therefore the order β^0 only contributes outside the surface layer, while the order β terms come from plaquettes on the surface layer. In this way we recover the well known lowest order strong coupling result $Z[J]/Z[0] \sim \beta^A$. More important, we have seen that indeed $f^{(min)}$ is the crucial quantity.

For a Wilson loop in the adjoint representation one does not expect an area law, because the charges can be screened by pair creation. This can be easily checked in our approach. The traces in the adjoint and in the fundamental representation are related by $TrW_{(1)} = [TrW_{(1/2)}]^2 - 1$. With our parametrization we obtain

$$\begin{aligned} [TrW_{(1/2)}]^2 &= \left[\sum_{\hat{n}} W_{\hat{n}} \exp[i \sum_{q\lambda b} J_\lambda(q) \hat{s}^b_{\lambda\hat{n}}(q) \Theta^b_\lambda(q)]\right]^2 \\ &= \sum_{\hat{n}\hat{n}'} W_{\hat{n}} W_{\hat{n}'} \exp\left[i \sum_{q\lambda b} J_\lambda(q) [\hat{s}^b_{\lambda\hat{n}}(q) + \hat{s}^b_{\lambda\hat{n}'}(q)] \Theta^b_\lambda(q)\right]. \end{aligned} \tag{5.4}$$

The last term on the rhs of the modified equation (3.12) becomes $J_\mu(r)[\hat{s}^a_{\mu\hat{n}}(q) + \hat{s}^a_{\mu\hat{n}'}(q)]$ and is always even. Therefore (4.2) becomes a homogeneous equation, no minimal layer and no area law will appear. Similarly one can see that we don't get confinement if we replace $SU(2)$ by $SO(3)$.

With some technical effort or a more streamlined approach it should be possible to carry through a similar analysis for $SU(3)$. It would be nice to see, how the formalism would create the expected Z_3 structure.

Our conclusions which led to the area law would break down if the result, by some reason whatsoever, would be independent of $f_{\rho\nu}(p)$, thereby giving a vanishing string tension. This appears hardly possible for a finite lattice. We have seen before that there is indeed an essential dependence on $f_{\rho\nu}(p)$ in the strong coupling limit $\beta \to 0$. Such a dependence must survive for all finite β because the original expression $Z[J]$ in (2.1) clearly fulfills the strict inequalities $0 < Z[J] < Z[0]$. The string tension might, however, vanish in a certain region of β after performing the thermodynamic limit. In particular such an effect could be expected in higher dimensions, where the presence of the two dimensional layer becomes relatively less important than in lower dimensions. Indeed it is known [6] that lattice $SU(2)$ has a first order phase transition for $d = 5$ at $\beta_c = 1.642 \pm 0.015$.

We come back to $d = 4$. At the end one is interested in the continuum limit $\beta \to \infty$ which requires a particular investigation. If the string tension is a physical quantity and β goes to infinity as prescribed by the renormalization group, a non vanishing string tension for the lattice theory will persist in the continuum limit.

6 Interpretation and conclusions

There is an extensive literature on the various pictures of confinement which cannot be discussed here. For a recent review we refer to [7]. We come directly to the physical interpretation of our results. The key is equation (4.2) for the Z_2 field strength tensor,

$$\sum_\nu \Delta_\nu f_{\mu\nu}(r) + J_\mu(r) = 0 \pmod 2. \tag{6.1}$$

The solutions in form of layers on two-dimensional surfaces were discussed in detail in sect. 4.

In d=3 dimensions put $f_{\mu\nu}(r) = \sum_\lambda \epsilon_{\mu\nu\lambda} B_\lambda(r) \pmod{\mu\nu\ 2}$. Then (6.1) becomes $\nabla \times \mathbf{B} = \mathbf{J} \pmod 2$. The magnetic field $\mathbf{B}$ has sources corresponding to magnetic monopoles. It is reasonable to use the right derivative in the divergence, and to associate the monopole density $\tilde{\rho}$ with cubes as usual. We therefore define

$$\tilde{\rho}(r_{123}) = \sum_\nu \Delta_\nu^{(right)} B_\nu(r) \pmod 2. \tag{6.2}$$

The solution $f_{\mu\nu}^{(min)}(r)$ then immediately leads to a double layer of monopoles in the cubes on both sides of the minimal surface.

For d=4 we define the dual tensor $\tilde{f}_{\mu\nu}(r) = (1/2) \sum_{\lambda\kappa} \epsilon_{\mu\nu\lambda\kappa} f_{\lambda\kappa}(r) \pmod{\mu\nu\ 2}$. The conserved (mod 2) magnetic current $\tilde{J}_\mu$ lives on 3-dimensional cubes $r_{\rho\lambda\kappa}$, where ρ, λ, κ denote the three directions orthogonal to μ.

$$\tilde{J}_\mu(r_{\rho\lambda\kappa}) = \sum_\nu \Delta_\nu^{(right)} \tilde{f}_{\mu\nu}(r) \pmod 2. \tag{6.3}$$

Consider a loop in the x_1-x_4-plane, with x_4 interpreted as euclidean time. Let x_1, x_4 be within the loop and suppress the x_4-extension of the cubes. For the solution $f_{\mu\nu}^{(min)}(r)$ we then get a non-vanishing $\tilde{J}_\mu$ on all plaquettes in the x_1-x_2-plane and in the x_1-x_3-plane

which contact the line x_2=x_3=0. We thus have a string of electric field $E_1(r) = f_{14}(r)$ in x_1-direction, concentrated on x_2=x_3=0. This is surrounded by magnetic current loops parallel to the x_2-x_3-plane which circle around the electric string. The configuration is therefore just dual to an Abrikosov vortex in a type II superconductor, where the magnetic field is encircled by the electric current. Flux quantization is evident, the Z_2-structure only allows for one unit of flux.

Configurations $f_{\mu\nu}(r)$ in (4.6) with $f_\kappa(r) \neq 0$ belong to other surfaces which are bounded by the loop as discussed in sect. 4. In addition closed surfaces can appear. The interpretation is similar as above. For illustration, connect e.g. two points in the x_1-x_2-plane by a path in form of a stair. Then $\tilde{J}_\mu$ lies on the plaquettes which point from the stair into positive and negative x_3-direction. All surfaces are summed with the appropriate weight in the partition function. A careful investigation of the various weights should give information about the extension of the electric flux tube.

We finally check the dual London equation, $\nabla \times \tilde{\mathbf{J}} = \frac{1}{\Lambda}\mathbf{E} \pmod 2$. It is easily seen that $(\nabla \times \tilde{\mathbf{J}})_\mu(r) \pmod 2$ is equal to 0 (1) if the link r_μ has contact to an even (odd) number of plaquettes with non-vanishing magnetic current. For the examples discussed above this means that $\nabla \times \tilde{\mathbf{J}}$ runs along the boundary of the set of plaquettes which carry the magnetic current. For the minimal layer, $\nabla \times \tilde{\mathbf{J}}$ is parallel to $\mathbf{E}$ as it should. It is, however, not concentrated on x_2=x_3=0 as the electric field, but on the four lines x_2=0, $x_3 = \pm 1$ and x_3=0, $x_2 = \pm 1$. For the stair, $\nabla \times \tilde{\mathbf{J}}$ is shifted by $x_3 = \pm 1$ with respect to $\mathbf{E}$. In general the dual London equation is essentially fulfilled, the two sides of the equation are just slightly shifted against each other. This might be interpreted by a non-vanishing Ginzburg-Landau coherence length $\tilde{\xi}$ which leads to a "normal" region near the string, where the London equation is not valid.

Let us compare with some familiar types of monopoles in the literature.

The charges of the $U(1)$ monopoles in [1] can take all integers, in obvious contrast to our Z_2 structure which only allows 0 and 1.

A popular definition of monopoles in $SU(2)$ is discussed e.g. in [8]. Let $\eta_{(p)} \equiv sign\, Tr\, U_{(p)}$ denote the sign of the plaquette action, and $\eta_c = \prod_{p\in\partial c} \eta_{(p)}$ the product of the $\eta_{(p)}$ around the boundary of the cube c. Then $\eta_c = -1$ represents a monopole in the (space like) cube c. There is a Z_2 structure as in our case.

Another frequently applied definition, reviewed e.g. in [9], uses the maximal abelian gauge. In a first step one maximizes the quantity $R = \sum_{r\mu} Tr[\sigma_3 U_\mu(r)\sigma_3 U_\mu^+(r)]$. The link matrices are then decomposed into a non-abelian and a $U(1)$ part, e.g. one can take the abelian link angle as the phase of $[U_\mu(r)]_{11}$. The $U(1)$ monopoles are then defined according to the DeGrand Toussaint construction [10] which allows monopole charges $0, \pm 1, \pm 2$.

The monopoles which naturally arose in the present work have no direct relation to any of these. An unconventional feature of our approach is the presence of monopoles in every configuration. In the approaches mentioned above there are plenty of configurations without any monopoles, namely those near the perturbative vacuum. In our case there is always a surface S bounded by the loop. This is associated with an electric string and accompanied by monopole vortices. From a physical point of view this appears quite attractive.

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A Appendix

For definiteness we give here the matrix S used by us in sect. 3 when selecting a convenient subset of the $\mathbf{S}_n$. It reads

$$S = \begin{vmatrix} -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} \tag{A.1}$$

Recall that the columns of S consist of 8 of the vectors $\mathbf{S}_n$, with the property that $\epsilon_n = +1$. The last component, corresponding to s_8, was fixed to 1.

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